Constructions for the Solution of the m Queens Problem

E. J. Hoffman; J. C. Loessi; R. C. Moore

Mathematics Magazine, Vol. 42, No. 2 (Mar., 1969), 66-72.

Stable URL:
http://links.jstor.org/sici?sici=0025-570X%28196903%2942%3A2%3C66%3ACFTSOT%3E2.0.CO%3B2-9

Mathematics Magazine is currently published by Mathematical Association of America.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/maa.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.

http://www.jstor.org/
Thu Dec 15 18:26:34 2005

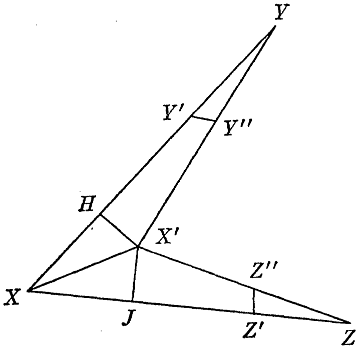

FIG. 11

Returning to polygons we see that if any five of the vertices of a polygon determine a nondegenerate conic, and lie on one branch of that conic, then the set of segment-generated points generated by the vertices is dense in a specifiable region of the plane. Also, the set of intersect-generated points is dense in the plane. Thus our doodling conjecture is decided.

CONSTRUCTIONS FOR THE SOLUTION OF THE m QUEENS PROBLEM

E. J. HOFFMAN, J. C. LOESSI, and R. C. MOORE,
The Johns Hopkins University Applied Physics Laboratory

The problem of the m queens, originally introduced by Gauss (with $m=8$), may be stated as follows: is it possible to place m queens on an $m \times m$ chessboard so that no one queen can be taken by any other? The problem is an interesting one because it reduces to that of finding a maximum internally stable set, S, of a symmetric graph, $G=(X, \Gamma)$, the vertices of which correspond to the m^2 square elements of an $m \times m$ matrix, where x' is an element of Γx only if x and x' are on the same row or column or diagonal, and where $\Gamma S \cap S$ is the null set. (See [1].) Obviously, $|S|$ cannot be greater than m.

By treating the chessboard as an $m \times m$ matrix of square elements, we can identify any square by an ordered pair, (i, j), where i and j are the row and column numbers of the square, respectively. We define a *major diagonal* of the matrix to be a set of elements (i, j) such that $m-j+i=$ CONSTANT where the CONSTANT is the number of the diagonal. The major diagonal numbered m will be called the *principal diagonal*. Clearly, all points on the principal diagonal have the property $i=j$.

We further define a *minor diagonal* of the matrix to be a set of elements (i, j) such that $i+j-1=$ CONSTANT where the CONSTANT is the number of the diagonal.

The m queens problem can now be stated as follows: place m queens on an $m \times m$ matrix of square elements so that, for the elements occupied,

a) the row numbers are unique,
b) the column numbers are unique,
c) the major diagonal numbers are unique, and
d) the minor diagonal numbers are unique.

The constructions which follow are sufficient to solve the m queens problem; the theorems delineate which of the constructions are appropriate for a given m. It will be shown that the solutions apply for all $m \geqq 4$.

Construction A. Form an $m \times m$ matrix of square elements with $m=2n$, where $n=2, 3, 4, 5, \cdots$.

i) Place queens on the elements (i_k, j_k), where $i_k=k$ and $j_k=2k$, $k=1, 2, 3, \cdots, n$.

ii) Place queens on the elements (i_l, j_l), where $i_l=2n+1-l$ and $i_l=2n+1-2l$, $l=1, 2, 3, \cdots, n$.

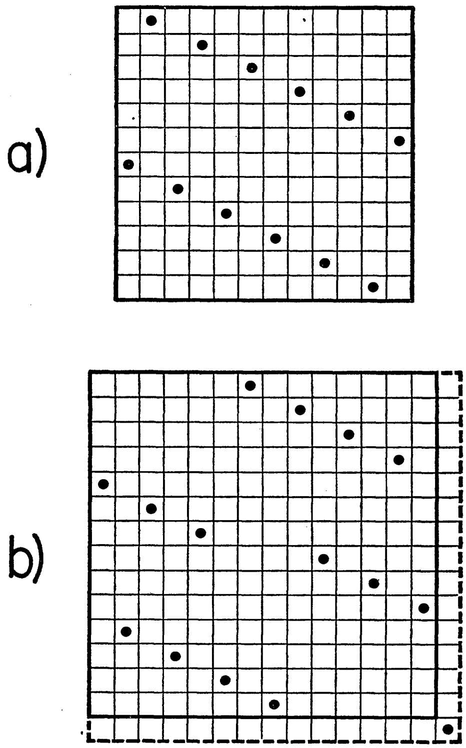

FIG. 1. (a) Solution of 12×12 matrix using Construction A. (b) Solution of 14×14 matrix using Construction B and extension to 15×15 matrix using Construction C.

Construction B. Form an $m \times m$ matrix of square elements with $m=2n$, where $n=2, 3, 4, 5, \cdots$.

i) Place queens on the elements (i_k, j_k), where $i_k = k$ and

$$j_k = 1 + \{[2(k-1) + n - 1] \text{ modulo } m\}, \qquad k = 1, 2, 3, \cdots, n.$$

ii) Place queens on the elements (i_l, j_l), where $i_l = 2n+1-l$ and $j_l = 2n - \{[2(l-1)+n-1] \text{ modulo } m\}$, $l = 1, 2, 3, \cdots, n$.

Construction C. To an $m \times m$ matrix of square elements add an $(m+1)$th row and an $(m+1)$th column. Place a queen on the element $(m+1, m+1)$.

Figure 1 shows typical examples of Constructions A, B, and C.

THEOREM 1. *A solution of the m queens problem is obtained when Construction A is applied to an* $m \times m$ *matrix,* $m = 2n$, *where n is an integer greater than zero such that* $n \neq 3\lambda + 1$, $\lambda = 0, 1, 2, \cdots$.

Proof. Part i) of Construction A places queens on the elements $(k, 2k)$ while part ii) places queens on the elements $(2n+1-l, 2n+1-2l)$, $1 \leqq (k, l) \leqq n$. Clearly, part i) places one queen on an element of each of the first n rows and also on each even-numbered column. Part ii) places one queen on an element of each of the second n rows and also on each odd-numbered column. Therefore, each row and column has one and only one queen.

The major diagonals which are used by part i) are numbered $2n - 2k + k = 2n - k$, $1 \leqq k \leqq n$. Clearly, these are unique. The major diagonals used by part ii) are numbered $2n - (2n+1-2l) + 2n + 1 - l = 2n + l$, $1 \leqq l \leqq n$. Clearly, these are also unique.

Assume a queen from part i) occupies the same major diagonal as a queen from part ii). Then $2n - k = 2n + l$ and $-k = l$ which is impossible, so we are forced to abandon the hypothesis that two queens occupy the same major diagonal.

The minor diagonals which are used by part i) are numbered $k + 2k - 1 = 3k - 1$, $1 \leqq k \leqq n$. Clearly, these are unique. The minor diagonals used by part ii) are numbered $2n+1-l+2n+1-2l-1 = 4n-3l+1$, $1 \leqq l \leqq n$. Clearly, these are also unique.

Assume a queen from part i) occupies the same minor diagonal as a queen from part ii) so that $3k - 1 = 4n - 3l + 1$ and $4n = 3(k+l) - 2$. Since n is an integer, $k + l$ must be even, and we can write

$$2n = 3\left(\frac{k+l}{2}\right) - 1.$$

Now we see that $(k+l)/2$ must be odd, say, $(k+l)/2 = 2\lambda + 1$, $\lambda = 0, 1, 2, \cdots$ and we have $2n = 3(2\lambda+1) - 1 = 6\lambda + 2$ so that $n = 3\lambda + 1$, $\lambda = 0, 1, 2, \cdots$, which are the values excluded by the theorem. Hence we are forced to abandon the hypothesis that two queens occupy the same minor diagonal, and the theorem is established.

THEOREM 2. *A solution of the m queens problem is obtained when Construction B is applied to an* $m \times m$ *matrix,* $m = 2n$, *where n is an integer greater than unity such that* $n \neq 3\lambda$, $\lambda = 1, 2, 3, \cdots$.

Proof. Part i) of Construction B places queens on the elements $(1, n)$, $(2, n+2)$, $(3, n+4)$, $\cdots$, (r, s) where

$$r = \begin{cases} \dfrac{n+2}{2}, & n \text{ even} \\ \dfrac{n+1}{2}, & n \text{ odd} \end{cases} \text{ and } s = \begin{cases} 2n, & n \text{ even} \\ 2n-1, & n \text{ odd} \end{cases},$$

and on the elements (r', s'), $(r'+1, s'+2)$, $\cdots$, $(n, n-2)$ where

$$r' = r+1 \quad \text{and} \quad s' = \begin{cases} 2, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}.$$

Part ii) of Construction B places queens on the elements $(2n, n+1)$, $(2n-1, n-1)$, $(2n-2, n-3)$, $\cdots$, (p, q) where

$$p = \begin{cases} \dfrac{3n}{2}, & n \text{ even} \\ \dfrac{3n+1}{2}, & n \text{ odd} \end{cases} \text{ and } q = \begin{cases} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{cases},$$

and on the elements (p', q'), $(p'-1, q'-2)$, $(p'-2, q'-4)$, $\cdots$, $(n+1, n+3)$ where

$$p' = p-1 \quad \text{and} \quad q' = \begin{cases} 2n-1, & n \text{ even} \\ 2n, & n \text{ odd} \end{cases}.$$

Clearly, part i) places one queen on an element of each of the first n rows, and also on each even-numbered column (if n is even) or each odd-numbered column (if n is odd). Part ii) places one queen on an element of each of the second n rows and also on each odd-numbered column (if n is even) or each even-numbered column (if n is odd). Therefore, each row and column has one and only one queen.

The major diagonals which are used by part i) are numbered $2n-[2(k-1)+n]+k=n-k+2$ for $1 \leqq k \leqq r$, and $2n-[2(k'-1)-n]+k'=3n-k'+2$ for $r' \leqq k' \leqq n$. Clearly, since the largest of the first set is $n-1+2=n+1$ and the smallest of the second set is $3n-n+2=2(n+1)$, these are unique. The major diagonals used by part ii) are numbered

$$2n - \{2n+1-[2(l-1)+n]\} + 2n+1-l = 3n+l-2 \qquad 1 \leqq l \leqq 2n+1-p,$$

and

$$2n - \{2n+1-[2(l'-1)-n]\} + 2n+1-l' = n+l'-2 \qquad 2n+1-p' \leqq l' \leqq n.$$

Clearly, since the smallest of the first set is $3n+1-2=3n-1$ and the largest of the second set is $n+n-2=2(n-1)$, these are unique.

Assume a queen from part i) occupies the same major diagonal as a queen from part ii). We would then have

(1) $n-k+2=3n+l-2$, (3) $3n-k'+2=3n+l-2$, or

(2) $n-k+2=n+l'-2$, (4) $3n-k'+2=n+l'-2$.

Equation (1) implies $k+l=4-2n$, but since the smallest $k+l$ can be is 2, this can never happen, since n is greater than unity. Equation (2) implies $k+l'=4$, and the smallest $k+l'$ will ever be is

$$\left\{\begin{matrix} \dfrac{n+6}{2}, & n \text{ even} \\ \dfrac{n+5}{2}, & n \text{ odd} \end{matrix}\right\}.$$

Equation (3) implies $k'+l=4$, and the smallest $k'+l$ will ever be is

$$\left\{\begin{matrix} \dfrac{n+6}{2}, & n \text{ even} \\ \dfrac{n+5}{2}, & n \text{ odd} \end{matrix}\right\}.$$

Equations (2) and (3) can be satisfied, therefore, only if $n=2$ or $n=3$. The value $n=3$ is excluded in the statement of the theorem. For $n=2$ we have $r=2$ and $2n+1-p=n$ so that $r'=r+1>n$ and $2n+1-p'=2n+2-p>n$, and neither k' nor l' can exist. Equation (4) implies $k'+l'=2n+4$, but the largest $k'+l'$ will ever be is $2n$, so we are forced to abandon the hypothesis that two queens occupy the same major diagonal.

The minor diagonals which are used by part i) are numbered $k+2(k-1)+n-1=n+3k-3$ for $1\leqq k\leqq r$, and $k'+2(k'-1)-n-1=-n+3k'-3$ for $r'\leqq k'\leqq n$. The minor diagonals used by part ii) are numbered $2n+1-l+2n+1-2(l-1)-n-1=3n-3l+3$ for $1\leqq l\leqq 2n+1-p$, and $2n+1-l'+2n+1-2(l'-1)+n-1=5n-3l'+3$ for $2n+1-p'\leqq l'\leqq n$.

Assume two queens occupy the same minor diagonal. We would then have

(5) $n+3k-3=-n+3k'-3$, (8) $-n+3k'-3=3n-3l+3$,

(6) $n+3k-3=3n-3l+3$, (9) $-n+3k'-3=5n-3l'+3$, or

(7) $n+3k-3=5n-3l'+3$, (10) $3n-3l+3=5n-3l'+3$.

Equation (5) implies $2n=3(k'-k)$, so $k'-k$ must be even, say $k'-k=2\lambda$, $\lambda=1, 2, 3, \cdots$. Then $2n=3(2\lambda)$, $n=3\lambda$, which are the values excluded by the theorem. Similarly, equation (10) results in $2n=3(l'-l)$, $l'-l=2\lambda$, $n=3\lambda$.

Equation (6) implies $2n=3(k+l-2)$, so $k+l-2$ must be even, say $k+l-2=2\lambda$, $\lambda=1, 2, 3, \cdots$. Then $2n=3(2\lambda)$, $n=3\lambda$, which are the values excluded by the theorem.

Equation (7) implies $4n=3(k+l'-2)$, so $k+l'-2$ must be doubly even, say $k+l'-2=4\lambda$, $\lambda=1, 2, 3, \cdots$. Then $4n=3(4\lambda)$, $n=3\lambda$, which are the values excluded by the theorem. Similarly, equation (8) results in $4n=3(k'+l-2)$, $k'+l-2=4\lambda$, $n=3\lambda$.

Finally, equation (9) implies $6n=3(k'+l'-2)$, $2n=k'+l'-2$. But the largest $k'+l'-2$ will ever be is $2n-2$, so we are forced to abandon the hypothesis that two queens occupy the same minor diagonal, and the theorem is established.

Before examining the validity of Construction C, it will be necessary to prove the following two lemmas:

LEMMA 1. *Construction* A *places no queens on the principal diagonal.*

Proof. The principal diagonal was defined as the major diagonal for which $i=j$. Suppose a queen from Construction A occupies the principal diagonal. Then either

(1) $k = 2k$ for some $1 \leqq k \leqq n$, or

(2) $2n+1-l = 2n+1-2l$ for some $1 \leqq l \leqq n$.

Equation (1) implies $k=0$, which contradicts the bound $k \geqq 1$. Likewise, equation (2) implies $l=0$, contradicting the bound $l \geqq 1$. We therefore abandon the hypothesis that a queen exists on the principal diagonal, and the lemma is established.

LEMMA 2. *Construction* B *places no queens on the principal diagonal.*

Proof. The reasoning here is similar to that in Lemma 1. Suppose a queen from part (i) of Construction B occupies the principal diagonal. Then either

(1) $2(k-1)+n = k$ for some $1 \leqq k \leqq r$, or

(2) $2(k'-1)-n = k'$ for some $r' \leqq k' \leqq n$.

Equation (1) implies $k=2-n$. But $n>1$, implying $k \leqq 0$, which contradicts the bound $k \geqq 1$. Equation (2) implies $k'=n+2$, contradicting the bound $k' \leqq n$. Thus no queen from part (i) can occupy the principal diagonal. Suppose a queen from part (ii) occupies the principal diagonal. Then either

(3) $2n+1-[2(l-1)+n] = 2n+1-l$ for some $1 \leqq l \leqq 2n+1-p$, or

(4) $2n+1-[2(l'-1)-n] = 2n+1-l'$ for some $2n+1-p' \leqq l' \leqq n$.

Equation (3) implies $l=2-n$. But $n>1$, implying $l \leqq 0$, which contradicts the bound $l \geqq 1$. Equation (4) implies $l'=n+2$, contradicting the bound $l' \leqq n$. Thus no queen from part (ii) can occupy the principal diagonal, and the lemma is established.

THEOREM 3. *A solution of the m queens problem for an $(m+1)\times(m+1)$ matrix is obtained when Construction* C *is applied to an $m\times m$ matrix which has previously been solved by Construction* A *or Construction* B.

Proof. By creating a new $(m+1)$th row and $(m+1)$th column and placing a queen at $(m+1, m+1)$, Construction C obviously preserves unique row and column numbers. In addition, it creates a new minor diagonal containing only the element $(m+1, m+1)$, thus preserving unique minor diagonal numbers.

The principal diagonal of the $(m+1)\times(m+1)$ matrix is, however, an extension of the principal diagonal of the $m\times m$ matrix. But from Lemmas 1 and 2 we know that this diagonal must have been empty. Hence, unique major diagonal numbers are preserved and the theorem is established.

Each of the three constructions contained an exclusion for certain values of m. It remains only to prove

THEOREM 4. *The m queens problem is solved for all $m\geqq 4$ by either Construction* A, B, *or* C.

Proof. Construction A applies to even m except for $m=2(3\lambda+1)$, $\lambda=0, 1, 2, \cdots$, or

(1) $$m = 6\lambda_A - 4 \qquad \lambda_A = 1, 2, 3, \cdots.$$

Construction B applies to even m except for $m=2(3\lambda_B)$

(2a) $$m = 6\lambda_B \qquad \lambda_B = 1, 2, 3, \cdots,$$

and

(2b) $$m = 2.$$

The special case, equation (2b), results from the exclusion of $n=1$ in the statement of Theorem 2.

Finally, Construction C applies to all odd m for which $m-1$ can be solved by either A or B.

Let M' be the set of integers $m'>1$ having the property that neither Construction A, B, or C solves the $m'\times m'$ matrix. Any even member, m_e', of M' must be simultaneously excluded from both A and B, implying either

(3) $$6\lambda_A - 4 = m_e' = 2, \quad \text{or}$$

(4a) $$6\lambda_A - 4 = m_e' = 6\lambda_B$$

(4b) $$\lambda_A - 2/3 = \lambda_B$$

for some pair of integers λ_A and λ_B. Equation (4) can never be satisfied by integer λ's. Equation (3) is satisfied only at $\lambda_A=1$, so that $m_e'=2$ is the only even member of M'.

Any odd member of M', say m_0', must be excluded from Construction C, implying that the even member $m_0'-1$ is excluded from both A and B. But we have seen that the only even number excluded from A and B is $m_e'=2$, so the only odd member of M' is $m_0'=3$. M' therefore contains only the two integers 2 and 3, and the theorem is established.

Reference

1. Claude Berge, The Theory of Graphs and its Applications, Wiley, New York, 1962, pp. 35–36.